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Modified Hadamard Product and Characterization Property by Analytic and Multivalent Functions

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Abstract

In this paper we introduce the subclass of uniformly convex and starlike functions which are analytic and multivalent with negative coefficients defined by using fractional calculus operators. Characterization property exhibited by the functions in the class and the results of modified Hadamard product are discussed. Connection with the popular subclasses like β-uniformly starlike, convex, pre-starlike, parabolic starlike and convex functions are also pointed out. Growth and distortion theorems, closure property, extreme points, class preserving integral operators, region of p-valency and other interesting properties of the class are also included.

Key Words and Phrases : *Multivalent functions, β-uniformly convex functions, Incomplete beta function, Modified Hadamard product, Parabolic starlike function.*

AMS Subject Classification : Primary 30C45, Secondary 26A33.

1. Introduction

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \quad (p, n \in I\!N) \tag{1.1}$$

which are analytic and multivalent in the open disc $E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also denote T_p, the subclass of A_p consisting of functions of the form

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k \quad (p, n \in I\!N; \quad a_k \geq 0). \tag{1.2}$$

A function $f(z) \in A_p$ is said to be β-uniformly starlike of order $\alpha, (-p \leq \alpha < p), \beta \geq 0$ and $z \in E$, denoted by $\beta - S(\alpha, p)$, if and only if

$$Re\left\{z\frac{f'(z)}{f(z)} - \alpha\right\} \geq \beta \left|z\frac{f'(z)}{f(z)} - p\right|. \tag{1.3}$$

A function $f(z) \in A_p$ is said to be β-uniformly convex of order $\alpha, (-p \leq \alpha < p), \beta \geq 0$ and $z \in E$, denoted by $\beta - K(\alpha, p)$, if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \geq \beta \left|1 + z\frac{f''(z)}{f'(z)} - p\right|. \tag{1.4}$$

Notice that, $\beta - S(\alpha, 0) = \beta - S(\alpha), \beta - K(\alpha, 0) = \beta - K(\alpha), 0 - S(\alpha) = S(\alpha)$ and $0 - K(\alpha) = K(\alpha)$, where $\beta - S(\alpha)$ and $\beta - K(\alpha)$ are the classes of β-uniformly starlike and β-uniformly convex functions of order $\alpha, (-1 \leq \alpha < 1).S(\alpha)$ and $K(\alpha)$ are the popular classes of starlike and convex functions of order $\alpha, (0 \leq \alpha < 1)$.

Obviously, $f \in \beta - K(\alpha, p)$ if and only if $zf' \in \beta - S(\alpha, p)$. The incomplete beta function $\phi_p(a, c; z)$ is defined by

$$\phi_p(a, c; z) = z^p + \sum_{k=p+n}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} z^k \tag{1.5}$$

for $a \in I\!R$ and $c \in I\!R \setminus \overline{z}_0$ where $\overline{z}_0 = \{0, -1, -2, \cdots\}, z \in E$. $(a)_k$ is the Pochhammer symbol defined by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1 & : \quad k = 0 \\ a(a+1)\cdots(a+k-1) & : \quad k \in I\!N \end{cases}$$

Next consider $L_p(a, c)$ which is motivated from Carlson - Shaffer operator [1] defined by

$$\begin{aligned} L_p(a, c)f(z) &= \phi_p(a, c; z) * f(z), \quad \text{for} \quad f \in A_p \\ &= z^p + \sum_{k=p+n}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} a_k z^k, \quad z \in E. \end{aligned} \tag{1.6}$$

Definition 1 : Let $\mu > 0$ and $\gamma, \eta \in I\!R$. Then in terms of the Gauss hypergeometric function ${}_2F_1$ the generalized fractional integral operator $I_{0,z}^{\mu,\gamma,\eta}$ of a function is defined by

$$I_{0,z}^{\mu,\gamma,\eta} f(z) = \frac{z^{-\mu-\gamma}}{\Gamma(\mu)} \int_0^z (z-t)^{\mu-1} f(t)\ {}_2F_1(\mu+\gamma, -\eta; \mu; 1-\frac{t}{z})dt$$

where the function $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin with order

$$f(z) = 0(|z|^{\epsilon}), \quad z \to 0 \tag{1.7}$$

for

$$\epsilon > \max\{0, \gamma - \eta\} - 1 \tag{1.8}$$

and the multiplicity of $(z-t)^{\mu-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

Definition 2 : Let $0 \le \mu < 1$ and $\gamma, \eta \in I\!R$. Then the generalized fractional derivative operator $J_{0,z}^{\mu,\gamma,\eta}$ of a function $f(z)$ is defined by

$$J_{0,z}^{\mu,\gamma,\eta} f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \left\{ z^{\mu-\gamma} \int_0^z (z-t)^{-\mu} f(t)\ {}_2F_1(\gamma-\mu, 1-\eta; 1-\mu; 1-\frac{t}{z})dt \right\} \tag{1.9}$$

where the function $f(z)$ is analytic in a simply-connected region of z-plane containing the origin, with the order as given in (1.7) and multiplicity of $(z-t)^{-\mu}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

Notice that

$$\begin{aligned} L_p(m+1,1)f(z) &= \frac{z^p}{(1-z)^{m+p}} * f(z) \\ &= z^p + \sum_{k=p+n}^{\infty} \frac{(m+1)_{k-p}}{(1)_{k-p}} a_k z^k \\ &= D^{m+p-1} f(z) \end{aligned} \tag{1.10}$$

is the Ruscheweyh derivative of order m. Also note that,

$$J_{0,z}^{\mu,\mu,\eta} f(z) = D_{0,z}^{\mu} f(z) \quad (0 \le \mu < 1) \tag{1.11}$$

is the fractional derivative operator of order μ. Consider

$$U_{0,z}^{\mu,\gamma,\eta} f(z) = \begin{cases} \frac{\Gamma(1+p-\gamma)\Gamma(1+p+\eta-\mu)}{\Gamma(1+p)\Gamma(1+p+\eta-\gamma)} z^{\gamma} J_{0,z}^{\mu,\gamma,\eta}; 0 \le \mu < 1 \\ \frac{\Gamma(1+p-\gamma)\Gamma(1+p+\eta-\mu)}{\Gamma(1+p)\Gamma(1+p+\eta-\gamma)} z^{\gamma} I_{0,z}^{-\mu,\gamma,\eta}; -\infty < \mu < 0 \end{cases} \tag{1.12}$$

Let

$$\begin{aligned} M_{0,z}^{\mu,\gamma,\eta} f(z) &= \phi_p(a,c,z) * U_{0,z}^{\mu,\gamma,\eta} f(z) \\ &= z^p + \sum_{k=p+n}^{\infty} \frac{(a)_{k-p}(1+p)_{k-p}(1+p+\eta-\gamma)_{k-p}}{(c)_{k-p}(1+p-\gamma)_{k-p}(1+p+\eta-\mu)_{k-p}} a_k z^k \end{aligned} \tag{1.13}$$

for $a, \in I\!R, c \in I\!R \setminus \overline{z}_0, \overline{z}_0 = \{0, -1, -2, \cdots\}$ (1.14)

Denote $S(\mu, \gamma, \eta, \alpha, \beta)$ subclass of functions $f \in A_p$ satisfying

$$Re \left\{ \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))'}{M_{0,z}^{\mu,\gamma,\eta} f(z)} - \alpha \right\} \geq \left| \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))'}{M_{0,z}^{\mu,\gamma,\eta} f(z)} - p \right| \tag{1.15}$$

$(-\infty < \mu < 1; -\infty < \gamma < 1; \eta \in I\!R^+; -p \leq \alpha < p; \beta \geq 0; a \in I\!R; c \in I\!R \setminus \overline{z}_0; z \in E)$
(1.16)

Let $K(\mu, \gamma, \eta, \alpha, \beta) = S(\mu, \gamma, \eta, \alpha, \beta) \cap T_p$. It is also interesting to note that $K(\mu, \gamma, \eta, \alpha, \beta)$ extends to the class of starlike, convex, β-uniformly starlike, β-uniformly convex, β-uniformly pre-starlike, parabolic starlike and convex functions for suitable choice of the parameter $a, c, \mu, \gamma, \eta, \alpha$ and β. For instance;

1. For $a = c; \mu = \gamma = 0$ the class reduces to $\beta - S(\alpha, p)$.

2. For $a = c; \mu = \gamma = 1$ the class reduces to $\beta - K(\alpha, p)$.

3. For $a = 2 - 2\alpha, c = 1; \mu = \gamma = 0$ the class reduces to β-prestarlike functions.

4. For $a = c, \mu = \gamma = 0, \alpha = 2\rho - 1, (0 \leq \rho < 1)$ the class reduces to parablic starlike of order ρ.

5. For $a = c, \mu = \gamma = 1, \alpha = 2\rho - 1, (0 \leq \rho < 1)$ the class reduces to parabolic convex of order ρ.

Several other classes studied by different authors can be derived from $K(\mu, \gamma, \eta, \alpha, \beta)$.

2. Coefficient Estimates

Theorem 2.1 : A function $f(z)$ defined by (1.1) is in $S(\mu,\gamma,\eta,\alpha,\beta)$ if

$$\sum_{k=p+n}^{\infty}[k(1+\beta)-(\alpha+p\beta)]g(k)|a_k|\le p-\alpha \tag{2.1}$$

with the limits for the parameters given in (1.16).

Proof : It suffices to show that

$$\beta\left|\frac{z(M_{0,z}^{\mu,\gamma,\eta}f(z))'}{M_{0,z}^{\mu,\gamma,\eta}f(z)}-p\right|-Re\left\{\frac{z(M_{0,z}^{\mu,\gamma,\eta}f(z))'}{M_{0,z}^{\mu,\gamma,\eta}f(z)}-p\right\}\le p-\alpha.$$

Notice that

$$\beta\left|\frac{z(M_{0,z}^{\mu,\gamma,\eta}f(z))'}{M_{0,z}^{\mu,\gamma,\eta}f(z)}-p\right|-Re\left\{\frac{z(M_{0,z}^{\mu,\gamma,\eta}f(z))'}{M_{0,z}^{\mu,\gamma,\eta}f(z)}-p\right\}$$

$$\le(1+\beta)\left|\frac{z(M_{0,z}^{\mu,\gamma,\eta}f(z))'}{M_{0,z}^{\mu,\gamma,\eta}f(z)}-p\right|\le\frac{(1+\beta)\sum\limits_{k=p+n}^{\infty}(k-p)g(k)|a_k|}{1-\sum\limits_{k=p}^{\infty}g(k)|a_k|}$$

where

$$g(k)=\frac{(a)_{k-p}(1+p)_{k-p}(1+p+\eta-\gamma)_{k-p}}{(c)_{k-p}(1+p-\gamma)_{k-p}(1+p+\eta-\mu)_{k-p}}. \tag{2.2}$$

The last inequality is bounded above by $(p-\alpha)$ if

$$\sum_{k=p+n}^{\infty}[k(1+\beta)-(\alpha+p\beta)]g(k)|a_k|\le p-\alpha.$$

This completes the proof.

Next, we state and prove the necessary and sufficient condition for $f(z)\in K(\mu,\gamma,\eta,\alpha,\beta)$.

Theorem 2.2 : A function $f(z)$ given by (1.2) is in the class $K(\mu,\gamma,\eta,\alpha,\beta)$, if and only if

$$\sum_{k=p+n}^{\infty}[k(1+\beta)-(\alpha+p\beta)]g(k)a_k\le p-\alpha \tag{2.3}$$

with limits for the parameters given by (1.16).

Proof : In view of Theorem 2.1, we need only to prove the sufficient part. Let $f(z)\in K(\mu,\gamma,\eta,\alpha,\beta)$ and z be real. Then by relation (1.15)

$$Re\left\{\frac{z(M_{0,z}^{\mu,\gamma,\eta}f(z))'}{M_{0,z}^{\mu,\gamma,\eta}f(z)}-\alpha\right\}\ge\beta\left|\frac{z(M_{0,z}^{\mu,\gamma,\eta}f(z))'}{M_{0,z}^{\mu,\gamma,\eta}f(z)}-p\right|$$

$$\frac{p-\sum\limits_{k=p+n}^{\infty}kg(k)a_kz^{k-p}}{1-\sum\limits_{k=p+n}^{\infty}g(k)a_kz^{k-p}}-\alpha\geq\beta\left|\frac{\sum\limits_{k=p+n}^{\infty}(k-p)g(k)a_kz^{k-p}}{1-\sum\limits_{k=p+n}^{\infty}g(k)a_kz^{k-p}}\right|.$$

Allowing $z \to 1$ along the real axis, we obtained the desired inequality. The result (2.2) is sharp for

$$f(z) = z^p - \frac{p-\alpha}{[k(1+\beta)-(\alpha+p\beta)]g(k)}z^{p+n}, n \in I\!N. \tag{2.4}$$

Corollary 1 : Let the function $f(z)$ defined by (1.2) be in the class $K(\mu,\gamma,\eta,\alpha,\beta)$. Then

$$a_k \leq \frac{p-\alpha}{[k(1+\beta)-(\alpha+p\beta)]g(k)} \qquad (k > p+n, n \in I\!N)$$

with equality for the function $f(z)$ given by

$$f(z) = z^p - \frac{p-\alpha}{[k(1+\beta)-(\alpha+p\beta)]g(k)}z^{p+n}; \qquad (n \in I\!N).$$

3. Connection with other Integral Operators

Theorem 3.1 : Let $\frac{a(1+p)(1+p+\eta-\gamma)}{c(1+p-\gamma)(1+p+\eta-\mu)} \leq 1$ for the limits of the parameters given by $(-\infty < \mu < 1; -\infty < \gamma < 1; \eta \in I\!R^+; -p \leq \alpha < p; \beta \geq 0; a \in I\!R; c \in I\!R \setminus \overline{z}_0; z \in E)$.

Also let the function $f(z)$ defined by (1.2) satisfy

$$\sum_{k=p+n}^{\infty}\frac{[k(1+\beta)-(\alpha+p\beta)]g(k)a_k}{p-\alpha} \leq \frac{c(1+p-\gamma)(1+p+\eta-\mu)}{a(1+p)(1+p+\eta-\gamma)}. \tag{3.1}$$

Then $M_{0,z}^{\mu,\gamma,\eta}f(z) \in K(\mu,\gamma,\eta,\alpha,\beta)$ where $g(k)$ is given by (2.2).

Proof : We have

$$\begin{aligned} M_{0,z}^{\mu,\gamma,\eta}f(z) &= z^p - \sum_{k=p+n}^{\infty}\frac{(a)_{k-p}(1+p)_{k-p}(1+p+\eta-\gamma)_{k-p}}{(c)_{k-p}(1+p-\gamma)_{k-p}(1+p+\eta-\mu)_{k-p}}a_kz^k \\ &= z^p - \sum_{k=p+n}^{\infty}g(k)a_kz^k \end{aligned} \tag{3.2}$$

where

$$g(k) = \frac{(a)_{k-p}(1+p)_{k-p}(1+p+\eta-\gamma)_{k-p}}{(c)_{k-p}(1+p-\gamma)_{k-p}(1+p+\eta-\mu)_{k-p}}. \tag{3.3}$$

Under the hypothesis of the theorem, we observe that the function $g(k)$ is a non-increasing function, that is, $g(p+n) \leq g(p+1)$ for all $n \in I\!N$. Thus,

$$0 < g(p+n) \leq g(p+1) = \frac{a(1+p)(1+p+\eta-\gamma)}{c(1+p-\gamma)(1+p+\eta-\mu)}. \tag{3.4}$$

Using (3.1) and (3.4), we get

$$\sum_{k=p+n}^{\infty} \frac{[k(1+\beta)-(\alpha+p\beta)]g^2(k)}{(p-\alpha)} a_k \leq g(2) \sum_{k=p+n}^{\infty} \frac{[k(1+\beta)-(\alpha+p\beta)]}{(p-\alpha)} g(k) \leq 1.$$

Therefore, by Theorem 2.2 we conclude that $M_{0,z}^{\mu,\gamma,\eta} f(z) \in K(\mu,\gamma,\eta,\alpha,\beta)$.

Remark : The equality in (3.1) is attained for the function

$$f(z) = z^p - \frac{c(p-\alpha)(1+p-\gamma)(1+p+\eta-\mu)}{a(1+p+\beta-\alpha)(1+p)(1+p+\eta-\gamma)} z^{p+1}. \tag{3.5}$$

Corollary 2 : Let $\mu, \gamma, \eta \in I\!R$ such that

$$\mu \geq 0, \gamma < 1+p, \max\{\mu,\gamma\} - (1+p) < \eta \leq \frac{\mu(\gamma-(2+p))}{\gamma} \tag{3.6}$$

also let the function $f(z)$ defined by (1.2) satisfy

$$\sum_{k=p+n}^{\infty} \frac{[k(1+\beta)-(\alpha+p\beta)]}{p-\alpha} a_k \leq \frac{(1+p-\gamma)(1+p+\eta-\mu)}{(1+p)(1+p+\eta-\gamma)} \tag{3.7}$$

for $-p \leq \alpha < p, \beta \geq 0$. Then $M_{0,z}^{\mu,\gamma,\eta} f(z) = J_{0,z}^{\mu,\gamma,\eta} f(z) \in \beta - S(\alpha,p)$.

Proof : The Corollary follows from Theorem 2.2 by setting $a = c$.

Remark : In Corollary 2 if $f(z)$ is given by (1.2) and $p = 1$ we get, corresponding result due to Jamal M. Shenan [6].

Corollary 3 : Let $\mu, \gamma, \eta \in I\!R$ such that

$$\mu \geq 0, \gamma < 1+p, \max\{\mu,\gamma\} - (1+p) < \eta \leq \frac{\mu(\gamma-(2+p))}{\gamma}$$

also let the function $f(z)$ defined by (1.2) satisfy

$$\sum_{k=p+n}^{\infty} \frac{k[k(1+\beta)-(\alpha+p\beta)]}{p-\alpha} a_k \leq \frac{c(1+p-\gamma)(1+p+\eta-\mu)}{a(1+p)(1+p+\eta-\gamma)} \tag{3.8}$$

for $-p \leq \alpha < p, \beta \geq 0, a = c$. Then $M_{0,z}^{\mu,\gamma,\eta} f(z) = J_{0,z}^{\mu,\gamma,\eta} f(z) \in \beta - K(\alpha,p)$.

Proof : The corollary follows from Theorem 2.2 by setting $a = c$.

Remark : In Corollary 3, if $f(z)$ is given by (1.2) and $p = 1$, we get the corresponding result due to Jamal M. Shenan [6].

Corollary 4 : Let $\mu = \gamma$ and η be real such that $-\infty < \mu < 1$, also let the function defined by (1.2) satisfy

$$\sum_{k=p+n}^{\infty} \frac{[k(1+\beta)-(\alpha+p\beta)]}{(p-\alpha)} a_k \leq \frac{c(1+p-\mu)}{a(1+p)} \tag{3.9}$$

for $-p \leq \alpha < p, \beta \geq 0$. Then $M_{0,z}^{\mu,\gamma,\eta} f(z) = D_{0,z}^{\mu} f(z) \in \beta - S(\alpha, p)$.

Corollary 5 : Let $\mu = \gamma$ and η be real such that $-\infty < \mu < 1$, also let the function defined by (1.2) satisfy

$$\sum_{k=p+n}^{\infty} \frac{k[k(1+\beta) - (\alpha + p\beta)]}{(p-\alpha)} a_k \leq \frac{(1+p-\mu)}{(1+p)} \tag{3.10}$$

for $-p \leq \alpha < p, \beta \geq 0$. Then $M_{0,z}^{\mu,\gamma,\eta} f(z) = D_{0,z}^{\mu} f(z) \in \beta - K(\alpha, p)$.

Proof : The corollary 4 and 5 follow from Theorem 2.2 by setting $\mu = \gamma, a = c$.

Corollary 6 : Let $\mu = \gamma = 0$ and η be real such that $a \in I\!R, c \in I\!R \setminus \overline{z}_0, \overline{z}_0 = \{0, -1, -2, \cdots\}$, also let the function defined by (1.2) satisfy

$$\sum_{k=p+n}^{\infty} \frac{[k(1+\beta) - (\alpha + p\beta)]}{p-\alpha} a_k \leq \frac{c}{a} \tag{3.11}$$

for $-p \leq \alpha < p, \beta \geq 0$. Then $M_{0,z}^{\mu,\gamma,\eta} f(z) = \phi_p(a, c; z) * f(z) \in \beta - S(\alpha, p)$.

Corollary 7 : Let $\mu = \gamma = 0$ and η be real such that $a \in I\!R \setminus \overline{z}_0, \overline{z}_0 = \{0, -1, -2, \cdots\}$, also let the function defined by (1.2) satisfy

$$\sum_{k=p+n}^{\infty} \frac{k[k(1+\beta) - (\alpha + p\beta)]}{p-\alpha} a_k \leq \frac{c}{a} \tag{3.12}$$

for $-p \leq \alpha < p, \beta \geq 0$. Then $M_{0,z}^{\mu,\gamma,\eta} f(z) = \phi_p(a, c; z) * f(z) \in \beta - K(\alpha, p)$.

Proof : The Corollary 6 and 7 follow from Theorem 2.2 by setting $\mu = \gamma = 0$.

4. Results on Modified Hadamard Product

Theorem 4.1 : Let the functions $f(z)$ and $g(z)$ defined by

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k \quad \text{and} \tag{4.1}$$

$$g(z) = z^p - \sum_{k=p+n}^{\infty} b_k z^k \tag{4.2}$$

belongs to $K(\mu, \gamma, \eta, \alpha, \beta)$ and $K(\mu, \gamma, \eta, \xi, \beta)$, respectively. Also assume that $\frac{a(1+p)(1+p+\eta-\gamma)}{c(1+p-\gamma)(1+p\eta-\mu)} \leq 1$. Then $(f * g)(z) \in K(\mu, \gamma, \eta, \delta, \beta)$ where

$$\delta = p - \frac{(1+\beta)(p-\alpha)(p-\xi)}{(1+p+\beta-\alpha)(1+p+\beta-\xi)g(p+1) - (p-\alpha)(p-\xi)} \tag{4.3}$$

and the result is sharp for

$$f(z) = z^p - \frac{(p-\alpha)}{(1+p+\beta-\alpha)g(p+1)} z^{p+1}$$

$$g(z) = z^p - \frac{(p-\xi)}{(1+p+\beta-\xi)g(p+1)} z^{p+1}.$$

Proof : To prove the theorem it is sufficient to show that

$$\sum_{k=p+n}^{\infty} \frac{[k(1+\beta)-(\alpha+p\beta)]}{(p-\delta)} g(k) a_k b_k \leq 1 \tag{4.4}$$

where $g(k)$ is defined by (3.3) and δ defined in (4.3).

Now, $f(z) \in K(\mu,\gamma,\eta,\alpha,\beta)$ and $g(z) \in K(\mu,\gamma,\eta,\xi,\beta)$ and thus, we have

$$\sum_{k=p+n}^{\infty} \frac{[k(1+\beta)-(\alpha+p\beta)]}{p-\alpha} g(k) a_k \leq 1 \tag{4.5}$$

$$\sum_{k=p+n}^{\infty} \frac{[k(1+\beta)-(\xi+p\beta)]}{p-\xi} g(k) b_k \leq 1. \tag{4.6}$$

By applying Cauchy-Schwarz inequality to (4.5) and (4.6) we get

$$\sum_{k=p+n}^{\infty} \frac{\sqrt{[(k(1+\beta)-(\alpha+p\beta)][k(1+\beta)-(\xi+p\beta)]}}{\sqrt{(p-\alpha)(p-\xi)}} g(k)\sqrt{a_k b_k} \leq 1. \tag{4.7}$$

In view of (4.4) it suffices to show that

$$\begin{aligned} &\sum_{k=p+n}^{\infty} \frac{[k(1+\beta)-(\delta+p\beta)]}{p-\delta} g(k) a_k b_k \\ &\leq \sum_{k=p+n}^{\infty} \frac{\sqrt{[k(1+\beta)-(\alpha+p\beta)][k(1+\beta)-(\xi+p\beta)]}}{\sqrt{(p-\alpha)(p-\xi)}} g(k)\sqrt{a_k b_k} \end{aligned}$$

or equivalently

$$\sqrt{a_k b_k} \leq \frac{\sqrt{[k(1+\beta)-(\alpha+p\beta)][k(1+\beta)-(\xi+p\beta)]}}{\sqrt{(p-\alpha)(p-\xi)}} \frac{(p-\delta)}{[k(1+\beta)-(\delta+p\beta)]} \text{ for } k \geq p+1. \tag{4.8}$$

In view of (4.7) and (4.8) it is enough to show that

$$\begin{aligned} &\frac{\sqrt{(p-\alpha)(p-\xi)}}{g(k)\sqrt{[k(1+\beta)-(\alpha+p\beta)][k(1+\beta)-(\xi+p\beta)]}} \\ &\leq \frac{\sqrt{[k(1+\beta)-(\alpha+p\beta)][k(1+\beta)-(\xi+p\beta)]}(p-\delta)}{\sqrt{(p-\alpha)(p-\xi)}[k(1+\beta)-(\delta+p\beta)]} \end{aligned}$$

which simplifies to

$$\delta \le p - \frac{(1+\beta)(k-p)(p-\alpha)(p-\xi)}{[k(1+\beta)-(\alpha+p\beta)][k(1+\beta)-(\xi+p\beta)]g(k)-(p-\alpha)(p-\xi)} \tag{4.9}$$

where

$$g(k) = \frac{(a)_{k-p}(1+p)_{k-p}(1+p+\eta-\gamma)_{k-p}}{(c)_{k-p}(1+p-\gamma)_{k-p}(1+p+\eta-\mu)_{k-p}} \quad \text{for} \quad k \ge p+1.$$

Notice that $g(k)$ is decreasing function of k $(k \ge p+1)$ and thus δ can be chosen as below.

$$\delta = p - \frac{(1+\beta)(p-\alpha)(p-\xi)}{(1+p+\beta-\alpha)(1+p+\beta-\xi)g(p+1)-(p-\alpha)(p-\xi)} \tag{4.10}$$

where

$$g(p+1) = \frac{a(1+p)(1+p+\eta-\gamma)}{c(1+p-\gamma)(1+p+\eta-\mu)}. \tag{4.11}$$

This completes the proof.

Theorem 4.2 : Let the function $f(z)$ and $g(z)$ defined as in Theorem 4.1 be in the class $K(\mu,\gamma,\eta,\alpha,b)$. Then $(f*g)(z) \in K(\mu,\gamma,\eta,\delta,\beta)$, where

$$\delta = p - \frac{(1+\beta)(p-\alpha)^2}{(1+p+\beta-\alpha)^2 g(p+1)-(p-\alpha)^2}$$

for $g(p+1)$ given by (4.11).

Proof : Substituting $\alpha = \xi$ in Theorem 4.1, the result follows.

Theorem 4.3 : Let the function $f(z)$ defined by (1.2) be in the class $K(\mu,\gamma,\eta,\alpha,\beta)$ and let $g(z) = z^p - \sum\limits_{k=p+n}^{\infty} b_k z^k$ for $|b_k| \le 1$. Then $(f*g)(z) \in K(\mu,\gamma,\eta,\alpha,\beta)$.

Proof : Notice that

$$\begin{aligned}
&\sum_{k=p+n}^{\infty} [k(1+\beta)-(\alpha+p\beta)]g(k)|a_k b_k| \\
&= \sum_{k=p+n}^{\infty} [k(1+\beta)-(\alpha+p\beta)]g(k)a_k|b_k| \\
&\le \sum_{k=p+n}^{\infty} [k(1+\beta)-(\alpha+p\beta)]g(k)a_k \\
&\le p-\alpha \quad \text{using Theorem 2.2.}
\end{aligned}$$

Hence $(f*g)(z) \in K(\mu,\gamma,\eta,\alpha,\beta)$.

Corollary 8 : Let the function $f(z)$ defined by (1.2) be in the class $K(\mu,\gamma,\eta,\alpha,\beta)$. Also let $(z) = z^p - \sum\limits_{k=p+n}^{\infty} b_k z^k$ for $0 \le b_k \le 1$. Then $(f*g)(z) \in K(\mu,\gamma,\eta,\alpha,\beta)$.

5. Inclusion Properties

In this Section we give the inclusion theorems for functions in the class $K(\mu,\gamma,\eta,\alpha,\beta)$.

Theorem 5.1 : Let the function $f(z)$ and $g(z)$ defined by (4.1) and (4.2) be in the class $K(\mu,\gamma,\eta,\alpha,\beta)$. Then the function $h(z)$ defined by

$$h(z) = z^p - \sum_{k=p+n}^{\infty} (a_k^2 + b_k^2) z^k$$

is in the class $K(\mu,\gamma,\eta,\theta,\beta)$ where

$$\theta = p - \frac{2(1+\beta)(p-\alpha)^2}{(1+p+\beta-\alpha)^2 g(p+1) - 2(p-\alpha)^2}$$

with $g(p+1)$ given by (4.11).

Proof : In view of Theorem 2.2 it is sufficient to show that

$$\sum_{k=p+n}^{\infty} \frac{[k(1+\beta)-(\theta+p\beta)]}{p-\theta} g(k)(a_k^2+b_k^2) \le 1. \tag{5.1}$$

Notice that $f(z)$ and $g(z)$ belong to $K(\mu,\gamma,\eta,\alpha,\beta)$ and so

$$\sum_{k=p+n}^{\infty} \left[\frac{(k(1+\beta)-(\alpha+p\beta))g(k)}{(p-\alpha)}\right]^2 a_k^2 \le \left[\sum_{k=p+n}^{\infty} \frac{(k(1+\beta)-(\alpha+p\beta))g(k)}{p-\alpha} a_k\right]^2 \le 1 \tag{5.2}$$

$$\sum_{k=p+n}^{\infty} \left[\frac{(k(1+\beta)-(\alpha+p\beta))g(k)}{(p-\alpha)}\right]^2 b_k^2 \le \left[\sum_{k=p+n}^{\infty} \frac{(k(1+\beta)-(\alpha+p\beta))g(k)}{p-\alpha} b_k\right]^2 \le 1. \tag{5.3}$$

Adding (5.2) and (5.3), we get

$$\sum_{k=p+n}^{\infty} \frac{1}{2} \left[\frac{(k(1+\beta)-(\alpha+p\beta))g(k)}{(p-\alpha)}\right]^2 (a_k^2+b_k^2) \le 1. \tag{5.4}$$

thus, (5.1) will hold if

$$\frac{k(1+\beta)-(\theta+p\beta))}{(p-\theta)} \le \frac{1}{2}\frac{g(k)[k(1+\beta)-(\alpha+p\beta))]^2}{(p-\alpha)^2}.$$

That is, if

$$\theta \leq p - \frac{2(1+\beta)(k-p)(p-\alpha)^2}{[k(1+\beta)-(\alpha+\beta)]^2 g(k) - 2(p-\alpha)^2}.$$

Notice that, θ can be further improved by using the fact that $g(p+n) \leq g(p+1)$ for $n \in \mathbb{N}$. Therefore,

$$\theta = p - \frac{2(1+\beta)(p-\alpha)^2}{(1+p+\beta-\alpha)^2 g(p+1) - 2(p-\alpha)^2}$$

where $g(p+1)$ is given by (4.11).

Theorem 5.2 : Let the function f and g belong to the class $K(\mu,\gamma,\eta,\alpha,\beta)$. Then for $\lambda \in [0,1]$, the function $h(z) = (1-\lambda)f(z) + \lambda g(z)$ is in the class $K(\mu,\gamma,\eta,\alpha,\beta)$.

Proof : Since $f(z)$ and $g(z)$ are in the class $K(\mu,\gamma,\eta,\alpha,\beta)$ they satisfy inequality (2.2). Therefore, $h(z)$ defined by

$$h(z) = (1-\lambda)f(z) + \lambda g(z) = z^p - \sum_{k=p+n}^{\infty} c_k z^k$$

where $c_k = (1-\lambda)a_k + \lambda b_k > 0$ is in the class $K(\mu,\gamma,\eta,\alpha,\beta)$.

Hence, $K(\mu,\gamma,\eta,\alpha,\beta)$ is indeed a convex set.

Theorem 5.3 : Let $f_j(z)$ defined as

$$f_j(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,j} z^k, \quad j = 1, 2, \cdots, \ell$$

belongs to the class $K(\mu,\gamma,\eta,\alpha,\beta)$. Then the function

$$h(z) = \frac{1}{\ell} \sum_{j=1}^{\ell} f_j(z)$$

is also in the class $K(\mu,\gamma,\eta,\alpha,\beta)$.

Proof : Since $f_j(z) \in K(\mu,\gamma,\eta,\alpha,\beta)$, in view of Theorem 2.2, we have

$$\sum_{k=p+n}^{\infty} \frac{[k(1+\beta)-(\alpha+p\beta)]g(k)}{(p-\alpha)} a_{k,j} \leq 1. \tag{5.5}$$

Now,

$$\frac{1}{\ell}\sum_{j=1}^{\ell} f_j(z) = z^p - \frac{1}{\ell}\sum_{j=1}^{\ell}\sum_{k=p+n}^{\infty} a_{k,j} z^k = z^p - \sum_{k=p+n}^{\infty} e_k z^k$$

where $e_k = \frac{1}{\ell}\sum\limits_{j=1}^{\ell} a_{k,j}$. Notice that

$$\sum_{k=p+n}^{\infty} \frac{[k(1+\beta)-(\alpha+p\beta)]g(k)}{(p-\alpha)} \frac{1}{\ell}\sum_{j=1}^{\ell} a_{k,j} \leq 1 \quad \text{using (5.5)}.$$

Thus, $h(z) \in K(\mu,\gamma,\eta,\alpha,\beta)$.

6. Extreme Points of the Class $K(\mu,\gamma,\eta,\alpha,\beta)$

Theorem 6.1 : Let $f_1(z) = z^p$ and

$$f_k(z) = z^p - \frac{(p-\alpha)}{[k(1+\beta)-(\alpha+p\beta)]g(k)} z^k, \quad (k \geq p+1).$$

Then $f(z) \in K(\mu,\gamma,\eta,\alpha,\beta)$ if and only if, $f(z)$ can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) \tag{6.1}$$

where $\lambda_k \geq 0$ and $\sum\limits_{k=1}^{\infty} \lambda_k = 1$.

Proof : Let $f(z)$ be expressible in the form

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z).$$

Then

$$f(z) = z^p - \sum_{k=2}^{\infty} \frac{(p-\alpha)}{[k(1+\beta)-(\alpha+p\beta)]g(k)} \lambda_k z^k.$$

Now,

$$\sum_{k=2}^{\infty} \frac{(p-\alpha)\lambda_k}{[k(1+\beta)-(\alpha+p\beta)]g(k)} \frac{[k(1+\beta)-(\alpha+p\beta)]g(k)}{(p-\alpha)} = \sum_{k=2}^{\infty} \lambda_k = 1-\lambda_1 \leq 1.$$

Therefore, $f(z) \in K(\mu,\gamma,\eta,\alpha,\beta)$.

Conversely, suppose that $f(z) \in K(\mu,\gamma,\eta,\alpha,\beta)$. Thus

$$a_k \leq \frac{p-\alpha}{[k(1+\beta)-(\alpha+p\beta)]g(k)}$$

Setting

$$\lambda_k = \frac{[k(1+\beta)-(\alpha+p\beta)]g(k)}{(p-\alpha)} a_k$$

and $\lambda_1 = 1 - \sum\limits_{k=2}^{\infty} \lambda_k$, we see that $f(z)$ can be expressed in the form (6.1).

Corollary 9 : The extreme points of the class $K(\mu, \gamma, \eta, \alpha, \beta)$ are $f_1(z) = z^p$ and

$$f_k(z) = z^p - \frac{p-\alpha}{[k(1+\beta)-(\alpha+p\beta)]g(k)} z^k, \quad k \geq p+1.$$

7. Growth and Distortion Theorems

Theorem 7.1 : Let the function $f(z)$ defined by (1.2) be in the class $K(\mu, \gamma, \eta, \alpha, \beta)$. Then

$$||M_{0,z}^{\mu,\gamma,\eta} f(z)| - |z|^p| \leq \frac{c(p-\alpha)(1+p-\gamma)(1+p+\eta-\mu)}{a(1+p)(1+p+\eta-\gamma)(1+p+\beta-\alpha)} |z|^{p+1} \quad \text{and} \tag{7.1}$$

$$||(M_{0,z}^{\mu,\gamma,\eta} f(z))'| - p|z|^{p-1}| \leq \frac{c(p-\alpha)(1+p-\gamma)(1+p+\eta-\mu)}{a(1+p+\eta-\gamma)(1+p+\beta-\alpha)} |z|^p. \tag{7.2}$$

Remark : The result (7.1) and (7.2) are sharp for the extremal function $f(z)$ given by

$$f(z) = z^p - \frac{c(p-\alpha)(1+p-\gamma)(1+p+\eta-\mu)}{a(1+p)(1+p+\eta-\gamma)(1+p+\beta-\alpha)} z^{p+1}. \tag{7.3}$$

Corollary 10 : Let $M_{0,z}^{\mu,\gamma,\eta} f(z) \in K(\mu, \gamma, \eta, \alpha, \beta)$ then the disc $|z| < 1$ is mapped onto a domain that contains the disc

$$|w| < 1 + \frac{c(p-\alpha)(1+p-\gamma)(1+p+\eta-\mu)}{a(1+p)(1+p+\eta-\gamma)(1+p+\beta-\alpha)}.$$

Also $(M_{0,z}^{\mu,\gamma,\eta} f(z))'$ maps the disc $|z| < 1$ onto a domain that contains the disc

$$|w| < p + \frac{c(p-\alpha)(1+p-\gamma)(1+p+\eta-\mu)}{a(1+p+\eta-\gamma)(1+p+\beta-\alpha)}$$

Remark : We can obtain the growth and distortion theorems for $J_{0,z}^{\mu,\gamma,\eta} f(z), D_{0,z}^{\mu} f(z)$ and $\phi_p(a, c, z)$ by accordingly initializing the parameters.

8. Family of Class Preserving Integral Operators

Here, we discuss some class preserving integral operators. Consider $F(z)$ defined by

$$F(z) = (J_{c,p} f)(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt \quad \text{for} \quad (f \in A_p; c > -p) \tag{8.1}$$

Let $G(z)$ be defined by

$$G(z) = z^{p-1} \int_0^z \frac{f(t)}{t^p} dt. \tag{8.2}$$

The Komatu operator [5] is defined by

$$H(z) = P^d_{c,p}f(z) = \frac{(c+p)^d}{\Gamma(d)z^c}\int_0^z t^{c-1}\left(\log\frac{z}{t}\right)^{d-1} f(t)dt \tag{8.3}$$

$(d > 0, c > -p, z \in E)$.

Another integral operator $I(z)$, which is the generalized Jun-Kim-Srivastava integral operator defined by

$$I(z) = Q^d_{c,p}f(z) = \binom{d+c+p-1}{c+p-1}\frac{d}{z^c}\int_0^z t^{c-1}(1-\frac{t}{z})^{d-1}f(t)dt \quad (d > 0, c > -p, z \in E) \tag{8.4}$$

Theorem 8.1 : Let $d > 0, c > -p$ and $f(z)$ belong to the class $K(\mu, \gamma, \eta, \alpha, \beta)$. Then the function $H(z)$ defined by (8.3) is p-valent in the disc $|z| < R_1$, where

$$R_1 = \inf_k \left\{\frac{[k(1+\beta)-(\alpha+p\beta)](c+k)^d g(k)}{k(c+p)^d(p-\alpha)}\right\}^{\frac{1}{k-p}}. \tag{8.5}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{k(c+p)^d(p-\alpha)}{[k(1+\beta)-(\alpha+p\beta)](c+k)^d g(k)}z^{p+n}, \quad n \in I\!N.$$

Proof : Notice that $H(z) \in K(\mu, \gamma, \eta, \alpha, \beta)$ and has the form

$$H(z) = z^p - \sum_{k=p+n}^{\infty}\left(\frac{c+p}{c+k}\right)^d a_k z^k. \tag{8.6}$$

In order to prove the assertion it is enough to show that

$$\left|\frac{H'(z)}{z^{p-1}} - p\right| \le p \ \text{ in } \ |z| < R_1. \tag{8.7}$$

Now,

$$\left|\frac{H'(z)}{z^{p-1}} - p\right| = \left|-\sum_{k=p+n}^{\infty} k\left(\frac{c+p}{c+k}\right)^d a_k z^{k-p}\right| \le \sum_{k=p+n}^{\infty} k\left(\frac{c+p}{c+k}\right)^d a_k|z|^{k-p}.$$

The last inequality is bounded above by p if

$$\sum_{k=p+n}^{\infty}\frac{k\left(\frac{c+p}{c+k}\right)^d a_k|z|^{k-p}}{p} \le 1. \tag{8.8}$$

Given that $f(z) \in K(\mu,\gamma,\eta,\alpha,\beta)$ and so, by Theorem 2.2

$$\sum_{k=p+n}^{\infty} \frac{[k(1+\beta)-(\alpha+p\beta)]}{p-\alpha} g(k) a_k \leq 1. \tag{8.9}$$

Thus inequality (8.8) will hold if

$$k\left(\frac{c+p}{c+k}\right)^d |z|^{k-p} \leq \frac{[k(1+\beta)-(\alpha+p\beta)]}{p-\alpha} g(k) \quad \text{for} \quad k \geq p+n.$$

That is, if

$$|z| \leq \left\{\frac{[k(1+\beta)-(\alpha+p\beta)](c+k)^d g(k)}{k(c+p)^d(p-\alpha)}\right\}^{\frac{1}{k-p}} \quad \text{for} \quad k \geq p+n, \ n \in \mathbb{N}.$$

The result follows by setting $|z| = R_1$.

Theorem 8.2 : Let $d > 0, c > -p$ and $f(z)$ belong to the class $K(\mu,\gamma,\eta,\alpha,\beta)$. Then the function $I(z)$ defined by (8.4) is p-valent in the disc $|z| < R_2$, where

$$R_2 = \inf_k \left\{\frac{[k(1+\beta)-(\alpha+p\beta)]\Gamma(d+c+k)\Gamma(c+p)g(k)}{k(p-\alpha)\Gamma(c+k)\Gamma(d+c+p)}\right\}^{\frac{1}{k-p}}.$$

The result is sharp for the function given by

$$f(z) = z^p - \frac{k(p-\alpha)\Gamma(c+k)\Gamma(d+c+p)}{[k(1+\beta)-(\alpha+p\beta)]\Gamma(d+c+k)\Gamma(c+p)g(k)} z^{p+n}, \quad n \in \mathbb{N}.$$

Proof : Notice that $I(z) \in K(\mu,\gamma,\eta,\alpha,\beta)$ and has the form

$$I(z) = z^p - \sum_{k=p+n}^{\infty} \frac{\Gamma(c+k)\Gamma(d+c+p)}{\Gamma(d+c+k)\Gamma(c+p)} a_k z^k.$$

Following arguments similar to those in Theorem 8.1 we get

$$|z| \leq \left\{\frac{[k(1+\beta)-(\alpha+p\beta)]\Gamma(d+c+k)\Gamma(c+p)g(k)}{k(p-\alpha)\Gamma(c+k)\Gamma(d+c+p)}\right\}^{\frac{1}{k-p}} \quad \text{for} \quad k \geq p+n, n \in \mathbb{N}.$$

9. Radius of Uniform Starlikeness, Convexity and Close-to -Convexity

Theorem 9.1 : Let the function $f(z)$ defined by (1.2) be in the class $K(\mu,\gamma,\eta,\alpha,\beta)$. Then $f(z)$ is p-valently starlike of order $s, (0 \leq s < p)$ in the disc $|z| \leq R_3$, where

$$R_3 = \inf_k \left\{\frac{(p-s)[k(1+\beta)-(\alpha+p\beta)]g(k)}{(k-s)(p-\alpha)}\right\}^{\frac{1}{k-p}}.$$

the result is sharp with the extremal function given by (2.4).

Proof : It is sufficient to show that

$$\left| z\frac{f'(z)}{f(z)} - 1 \right| \leq 1 - s \quad \text{for} \quad 0 \leq s < p$$

and $|z| < R_3$. With fairly straightforward calculation we can easily show that

$$|z| \leq \left\{ \frac{(p-s)[k(1+\beta) - (\alpha + p\beta)]g(k)]}{(k-s)(p-\alpha)} \right\}^{\frac{1}{k-p}}.$$

Setting $|z| = R_3$ we get the desired result.

Next, we state the theorems for radius of convexity and close-to-convexity.

Theorem 9.2 : Let the function $f(z)$ defined by (1.2) be in the class $K(\mu, \gamma, \eta, \alpha, \beta)$. Then $f(z)$ is p-valently convex of order $s, (0 \leq s < p)$ in the disc $|z| \leq R_4$ where

$$R_4 = \inf_k \left\{ \frac{p(p-s)[k(1+\beta) - (\alpha + p\beta)]g(k)}{k(k-s)(p-\alpha)} \right\}^{\frac{1}{k-p}}.$$

The result is sharp with the extremal function given by (2.4).

Theorem 9.3 : Let the function $f(z)$ defined by (1.20 be in the class $K(\mu, \gamma, \eta, \alpha, \beta)$. Then $f(z)$ is p-valently close-to-convex of order $s, (0 \leq s < p)$ in the disc $|z| \leq R_5$ where

$$R_5 = \inf_k \left\{ \frac{(p-s)[k(1+\beta) - (\alpha + p\beta)]g(k)]}{k(p-\alpha)} \right\}^{\frac{1}{k-p}}.$$

The result is sharp for the extremal function given by (2.4).

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